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ANALYTICITY AND A FINITE-ENERGY SUM RULE FOR THE REGGEON-PARTICLE AMPLITUDE IN a + b → c + d + e

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Abstract: We consider a single-Regge limit of the amplitude for the process a + b → c + d + e. In this limit the amplitude is proportional to the reggeon-particle amplitude for a + i → c + d, where i is a reggeon. We study the analytic structure of this amplitude using the dual resonance model and a perturbation theory model. We conclude that finite-energy sum rules can be derived, which relate the absorptive part of the amplitude at low $(p_c + p_d)^2$ to a part of the double-Regge vertex function of the original five-point amplitude. We discuss some phenomenological applications of the sum rules.

1. Introduction

In this paper we shall investigate the structure of the amplitude for the process a + b → c + d + e, where a, b, c, d and e are scalar particles, in the high-energy limit where $s_{ab} \to \infty$, $s_{dc} \to \infty$ while s_{cd} is kept fixed***, fig. 1. In such a limit the five-point amplitude is proportional to the reggeon-particle amplitude for a + i → c + d, where i is the exchanged reggeon. Although the singularity structure of the five-point function is rather complicated in general [1, 2] †, one may hope that its structure is simpler in a high-energy limit like that of fig. 1. This would then make possible the derivation of finite-energy sum rules [3; 4], which would be useful in the analysis of data for three-particle final states.

As was recently shown, there is an analogous situation in the case of inclusive reactions. The cross section for a + b → c + X is given by a discontinuity of the

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*** We use the notation $s_{ab} = (p_a + p_b)^2$, $s_{a\bar{c}} = (p_a - p_c)^2$, etc.

† In ref. [2] the problem of formulating finite-energy sum rules for five point amplitudes that relate the low-energy region in s_{ab} to the single-Regge limit is considered. This paper also contains a discussion of the general analyticity structure of the amplitude.

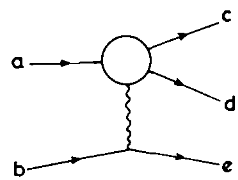

Fig. 1. The single-Regge limit of the process a + b → c + d + e.

amplitude for a + b + $\bar{c}$ → a + b + $\bar{c}$ in the forward direction [5], fig. 2a. In the limit when the missing mass is much smaller than the total energy, i.e. $s_{ab}/s_{ab\bar{c}} \gg 1$, the six-point amplitude is proportional to a reggeon-particle elastic amplitude (fig. 2b). In dual and ladder diagram models it turns out that the reggeon-particle amplitude has the singularity structure of a normal four-point function. Hence one may write down finite-mass sum rules [6], which connect the low missing-mass region with the triple-Regge limit. First applications of the FMSR to inclusive data are quite encouraging [7–9].

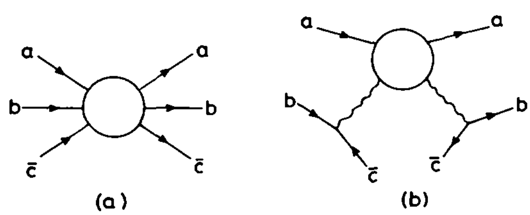

Fig. 2. (a) The six-point amplitude which is related to the inclusive reaction a + b → c + X. (b) A high-energy limit ($s_{ab}/s_{ab\bar{c}} \to \infty$ with $s_{b\bar{c}}$ fixed) of the amplitude in fig. 2a.

The reggeon-particle amplitude that we shall be concerned with here (fig. 1) is somewhat more general than the one encountered in inclusive distributions (fig. 2b). In the case of fig. 2b there is only one helicity amplitude contributing to the leading term, namely the one corresponding to a maximum helicity flip of the reggeons [10]. By contrast, there are many helicity states of the reggeon contributing [11] in fig. 1. The dependence on the helicity in this case can alternatively be seen as a dependence on the variable $\kappa = s_{cd}s_{de}/s_{ab}$. The reggeon-particle amplitude in fig. 1 also depends on the momentum transfer $s_{a\bar{c}}$. In the case of the inclusive reaction in fig. 2 the corresponding variables are equal to zero.

The structure of the amplitude for a + b → c + d + e in the double-Regge limit ($s_{cd} \to \infty$ in fig. 1) is already well-known [10, 11]. In this limit the amplitude decomposes into a sum of two terms, with cuts in s_{cd} and in s_{de}, respectively. We shall in the following be concerned only with a part of the five-point amplitude which in the double-Regge limit gives the first term (with a cut in s_{cd}). This is also the part which contains the poles in $s_{b\bar{e}}$ (when the reggeon is on-shell) and, according to the Steinmann relations [12], the normal threshold singularities and resonances in s_{cd}.

We investigate the singularity structure of this part of the amplitude in the single-Regge limit (s_{cd} finite) using the dual resonance model (DRM) and a perturbation theory model. We find that when $\kappa = 0$ the reggeon-particle amplitude has the singularity structure of a normal four-point amplitude.

The FESR which follow from this analytic structure relate the absorptive part of the low s_{cd} region to the first part of the double-Regge vertex. In the same way one can relate the other part of the double-Regge vertex to the low s_{de} region.

When $\kappa \neq 0$ the reggeon-particle amplitude has singularities which are not present in normal four-point amplitudes. These singularities do not, however, contribute to the leading term in the discontinuity as $s_{cd} \to \infty$. The effect of the new singularities in the FESR is therefore that of an additional term which is independent of the cut-off.

In sect. 2 we discuss the structure of the amplitude in the double Regge limit. The single Regge limit is considered in sect. 3, where we investigate the analytic structure of two models, the dual resonance model and a perturbation theory model. The structure of the two models turns out to be very similar. In sect. 4 we discuss the modifications due to left-hand singularities and signature. All essential properties found in sect. 3 remain unaltered for the signatured amplitudes. The FESR are derived in sect. 5 and some applications are considered in sect. 6.

2. The double-Regge limit

We shall begin our investigation of the analytic structure of the five-point amplitude by considering the double-Regge limit (fig. 3). This is defined by letting $s_{ab}, s_{cd}, s_{de} \to \infty$ keeping $s_{a\bar{c}}$, $s_{b\bar{e}}$ and $\kappa = s_{cd}s_{de}/s_{ab}$ fixed. The structure of the amplitude in this limit has been investigated by several authors [10, 11, 13]. It has been shown that an amplitude with only right-hand cuts in the asymptotic variables takes the form*

$$\begin{aligned} T = {} & (-s_{ab})^{\alpha_{b\bar{e}}}(-s_{cd})^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}}\, V_1(s_{b\bar{e}}, s_{a\bar{c}}\,; \kappa) \\ & + (-s_{ab})^{\alpha_{a\bar{c}}}(-s_{de})^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}}\, V_2(s_{a\bar{c}}, s_{b\bar{e}}\,; \kappa)\,, \end{aligned} \tag{1}$$

where $\alpha_{b\bar{e}} \equiv \alpha(s_{b\bar{e}})$, etc. The vertex functions V_1 and V_2 are entire functions of κ.

The important feature of the decomposition of T in eq. (1) is that only the first (second) term has a discontinuity in s_{cd} (s_{de}). If, according to duality, the Regge terms in eq. (1) are "built up" from resonances in s_{cd} and s_{de} we therefore

* This is true for amplitudes corresponding to planar Feynman diagrams and for planar dual models. For an example of the structure of a non-planar model see ref. [14].

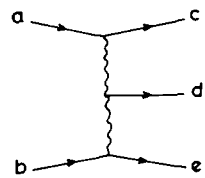

Fig. 3. The double Regge limit of the process a + b → c + d + e.

expect the first term in eq. (1) to be connected with the resonances in s_{cd} and the second term with resonances in s_{de}. In fact, the residue of a resonance in s_{cd} is a polynomial in s_{de} and thus cannot contribute to the discontinuity in s_{de} (i.e. to the second term in eq. (1)). The first term in eq. (1) also contains the poles in $\alpha_{b\bar{e}}$ when the reggeon i goes on-shell. From the point of view of duality in the system a + i → c + d we therefore should consider only a part of the five-point amplitude T, which in the double-Regge limit gives the first term in eq. (1).

In deriving the FESR we shall start from a dispersion relation in s_{cd} keeping s_{ab}, $s_{a\bar{c}}$, $s_{b\bar{e}}$ and κ fixed (note that the limit $s_{ab} \to \infty$ has already been taken as in fig. 1). The reason for keeping κ fixed is that we want the high-energy limit of the reggeon-particle amplitude ($s_{cd} \to \infty$) to be related to the double-Regge limit of the five-point function.

The variable s_{de} can be expressed in terms of the independent variables as

$$s_{de} = \frac{\kappa s_{ab}}{s_{cd}} \; ; \tag{2}$$

substituting eq. (2) into the expression (1) for T we get

$$T = (-s_{ab})^{\alpha_{b\bar{e}}} (-s_{cd})^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}} [V_1(s_{b\bar{e}}, s_{a\bar{c}}; \kappa) + (-\kappa)^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}} V_2(s_{a\bar{c}}, s_{b\bar{e}}; \kappa)] . \tag{3}$$

Both terms in eq. (3) now have a cut in s_{cd}, due to the relation (2). However, as discussed above only the first term can be dual to resonances in s_{cd}. The second term can be eliminated in either of two ways:

(i) By extrapolating to $\kappa = 0$. If $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} > 0$ the second term in eq. (3) vanishes. As we shall see below the situation is analogous in the single-Regge limit. Thus at $\kappa = 0$ the reggeon-particle amplitude has only normal four-point singularities in s_{cd} and a FESR can be derived. The FESR can be continued to $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} < 0$ by subtracting out the term which is singular when $\kappa \to 0$.

(ii) By considering the amplitude $\tilde{T}$:

$$\tilde{T} = \frac{1}{2i \sin \pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} [e^{i\pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} T(s_{de} + i\epsilon) - e^{-i\pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} T(s_{de} - i\epsilon)] . \tag{4}$$

In eq. (4), $T(s_{\mathrm{de}} \pm i\epsilon)$ is the amplitude obtained in the single-Regge limit letting $s_{\mathrm{de}} \to \infty$ above $(+i\epsilon)$ or below $(-i\epsilon)$ its cut. All other variables are to be evaluated in their physical limits. It follows from eq. (1) or eq. (3) that in the double-Regge limit

$$\tilde{T} = (-s_{\mathrm{ab}})^{\alpha_{\mathrm{b\bar{e}}}} (-s_{\mathrm{cd}})^{\alpha_{\mathrm{a\bar{c}}} - \alpha_{\mathrm{b\bar{e}}}} V_1(s_{\mathrm{b\bar{e}}}, s_{\mathrm{a\bar{c}}}; \kappa) . \tag{5}$$

In the next section we consider the singularity structure of $\tilde{T}$ in the single Regge limit. It turns out that $\tilde{T}$ has certain singularities in s_{cd} which are not present in normal four-point amplitudes. These come from the term $T(s_{\mathrm{de}} - i\epsilon)$ in eq. (4), where the amplitude T is evaluated in an unphysical limit. Such singularities cannot be determined from experimental data.

The additional singularities do not, however, contribute to the leading term (5) of $\tilde{T}$ in the double-Regge limit. This term is built up completely by the ordinary singularities in s_{cd}: The only effect of the extra singularities in the FESR is therefore to introduce a constant (i.e. cut-off independent) term on the l.h.s. of the sum rule.

This term vanishes in the limit $\kappa \to 0$, so that consistency with (i) is achieved.

3. The single-Regge limit

In this section we shall discuss the properties of the DRM and a perturbation theory model in the single-Regge limit shown in fig. 1. We consider amplitudes with only right-hand singularities, signature being introduced in the next section.

3.1. The dual resonance model

In the single-Regge limit of fig. 1 the B_5 amplitude can be expressed in terms of the hypergeometric function* $F(a, b; c; z)$

$$\begin{aligned} B_5 = \Gamma(-\alpha_{\mathrm{b\bar{e}}})(-\alpha_{\mathrm{ab}})^{\alpha_{\mathrm{b\bar{e}}}} & [B_4(-\alpha_{\mathrm{cd}}, -\alpha_{\mathrm{a\bar{c}}} + \alpha_{\mathrm{b\bar{e}}}) \\ & \times F\left(-\alpha_{\mathrm{b\bar{e}}}, -\alpha_{\mathrm{cd}}; 1 - \alpha_{\mathrm{b\bar{e}}} + \alpha_{\mathrm{a\bar{c}}}; \frac{\kappa}{s_{\mathrm{cd}}}\right) + \left(\frac{\kappa}{s_{\mathrm{cd}}}\right)^{\alpha_{\mathrm{b\bar{e}}} - \alpha_{\mathrm{a\bar{c}}}} \\ & \times B_4(-\alpha_{\mathrm{a\bar{c}}}, -\alpha_{\mathrm{b\bar{e}}} + \alpha_{\mathrm{a\bar{c}}}) F(-\alpha_{\mathrm{a\bar{c}}}, -\alpha_{\mathrm{cd}} - \alpha_{\mathrm{a\bar{c}}} + \alpha_{\mathrm{b\bar{e}}}; 1 - \alpha_{\mathrm{a\bar{c}}} \\ & + \alpha_{\mathrm{b\bar{e}}}; \frac{\kappa}{s_{\mathrm{cd}}})] . \end{aligned} \tag{6}$$

From this expression we can see the following.

(i) If $\kappa = 0$ and $\alpha_{\mathrm{b\bar{e}}} - \alpha_{\mathrm{a\bar{c}}} > 0$,

$$B_5 = \Gamma(-\alpha_{\mathrm{b\bar{e}}})(-\alpha_{\mathrm{ab}})^{\alpha_{\mathrm{b\bar{e}}}} B_4(-\alpha_{\mathrm{cd}}, -\alpha_{\mathrm{a\bar{c}}} + \alpha_{\mathrm{b\bar{e}}}) . \tag{7}$$

* See ref. [15]. The definition of the hypergeometric function is given in ref. [16].

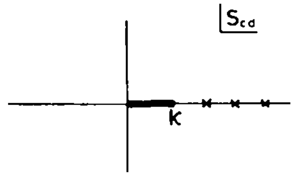

Fig. 4. The singularity structure in s_{cd} of the amplitude $\tilde{T}$ in the dual resonance model. The crosses correspond to poles and the thick line to a cut.

In this case the reggeon-particle amplitude is given by a B_4 function. The derivation of the FESR can be done as for a four a four-point function. If $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} < 0$ but, say, $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} > -1$, we can consider the amplitude

$$B_5' = B_5 - \left(\frac{\kappa}{s_{cd}}\right)^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}} B_4(-\alpha_{ac}, -\alpha_{b\bar{e}} + \alpha_{ac})\,\Gamma(\alpha_{b\bar{e}})(-\alpha_{ab})^{\alpha_{b\bar{e}}} \tag{8}$$

As $\kappa \to 0$ we find that B_5' reduces to a B_4 function as in eq. (7). The FESR which were derived for $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} > 0$ can thus be continued to $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} < 0$ by substracting out the terms which are singular when $\kappa \to 0$. In the double-Regge limit (3) this means that only the first (V_1) of the two terms is present. The FESR are therefore going to relate an integral over the absorptive part in s_{cd} to the first term in the double-Regge limit, as we already anticipated above.

(ii) When $\kappa \neq 0$ we can use the definition (4) to calculate $\tilde{T}$. The expression (6) for T is real when the variables α_{ab} and α_{de} are negative. Continuing α_{de} to positive values, using eq. (2) and the $\pm i\epsilon$ prescription at the branch point $\alpha_{de} = 0$, we get

$$\tilde{T} = \Gamma(-\alpha_{b\bar{e}})(-\alpha_{ab})^{\alpha_{b\bar{e}}} B_4(-\alpha_{cd}, -\alpha_{a\bar{c}} + \alpha_{be})$$
$$\times F\left(-\alpha_{b\bar{e}}, -\alpha_{cd}; 1 - \alpha_{be} + \alpha_{a\bar{c}}; \frac{\kappa}{s_{cd}}\right). \tag{9}$$

From eq. (9) one can directly see the singularity structure of $\tilde{T}$ in s_{cd} (fig. 4). There is a series of poles corresponding to resonances at $\alpha_{cd} = n$, $n = 0, 1, 2, \ldots$. In addition the F-function gives rise to a cut $0 \leqslant s_{cd} \leqslant \kappa$. This cut corresponds to a singularity of B_5 on an unphysical sheet in s_{de}. Thus it cannot be determined directly from experimental data. However, we may still derive a useful FESR from the singularity structure of fig. 4. The cut $0 \leqslant s_{cd} \leqslant \kappa$ gives in the FESR rise to a term which is independent of the cut-off. It can therefore be eliminated by varying the cut-off.

3.2. The perturbation theory model

Consider the five-point function T generated by a sum of Feynman diagrams in

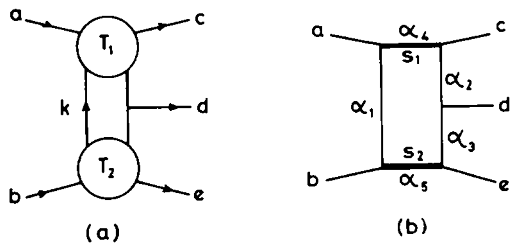

Fig. 5. (a) Diagrammatic representation of the perturbation theory model considered in the text. The internal lines correspond to scalar particles and the blobs T_1 and T_2 represent sums of planar Feynman diagrams. (b) The same amplitude as in fig. 5a; the thick lines indicate the energies which are dispersed in.

ϕ^3 theory*. We may describe the amplitude by the diagram in fig. 5a, where the blobs T_1 and T_2 represent sums of planar Feyman diagrams. We assume that the amplitudes T_1 and T_2 satisfy unsubtracted dispersion relations:

$$T_1((p_a+k)^2, s_{a\bar{c}}) = \int_0^\infty \frac{\sigma_1(s_1, s_{a\bar{c}})\, ds_1}{(p_a+k)^2 - s_1}\,, \tag{10}$$

and similarly for the amplitude T_2. We shall also assume that T_1 and T_2 are Regge behaved at high energy. Thus as $s_1 \to \infty$,

$$\sigma_1(s_1, s_{a\bar{c}}) \simeq \beta_1(s_{a\bar{c}})\, s_1^{\alpha_{a\bar{c}}}\,, \tag{11}$$

with a similar relation for σ_2 when $s_2 \to \infty$.

The five-point amplitude T of fig. 5a can now be expressed as (we take the mass of the propagating particles to be μ)

$$T = -ig\int \{\sigma_1(s_1, s_{a\bar{c}})\,\sigma_2(s_2, s_{b\bar{e}})\, ds_1 ds_2 d^4k\}\{[k^2-\mu^2]\,[(k+p_a-p_c)^2 - \mu^2]\,[(k+p_e-p_b)^2-\mu^2]\,[(k+p_a)^2 - s_1]\,[(k-p_b)^2 - s_2]\}^{-1}\,. \tag{12}$$

Apart from the integrations over s_1 and s_2, T has the structure of a simple box diagram (fig. 5b). Converting to the α-representation [1], the integration over the loop momentum can be done. We get then

$$T = 2g\pi^2 \int \frac{\sigma_1(s_1, s_{a\bar{c}})\,\sigma_2(s_2, s_{b\bar{e}})\, ds_1 ds_2 \prod_{i=1}^{5} d\alpha_i\, \delta\left(\sum_{i=1}^{5}\alpha_i - 1\right)}{[d+i\epsilon]^3}\,, \tag{13}$$

* For a review of the high-energy behaviour of Feynman diagrams, see ref. [1]. Models similar to that presented here have been studied by, for example, Drummond et al. [11] and Sanda [6].

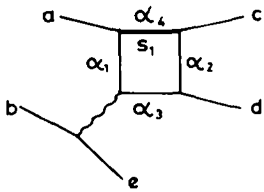

Fig. 6. The single-Regge limit of the amplitude shown in fig. 5.

where

$$d = \alpha_1\alpha_2 s_{a\bar{c}} + \alpha_1\alpha_3 s_{b\bar{e}} + \alpha_3\alpha_4 s_{cd} + \alpha_2\alpha_5 s_{de} + \alpha_4\alpha_5 s_{ab}$$
$$+ \alpha_1\alpha_4 m_a^2 + \alpha_2\alpha_4 m_c^2 + \alpha_2\alpha_3 m_d^2 + \alpha_3\alpha_5 m_e^2 + \alpha_1\alpha_5 m_b^2$$
$$- (\alpha_1 + \alpha_2 + \alpha_3)\mu^2 - \alpha_4 s_1 - \alpha_5 s_2 \,. \tag{14}$$

To find the structure of T in the single Regge limit of fig. 1 we let $s_{ab} \to -\infty$ and $s_{de} \to -\infty$ keeping s_{de}/s_{ab} fixed. When $-1 < \alpha_{b\bar{e}} < 0$ the leading contribution to the integral in eq. (13) comes from large s_2. Substituting the Regge expression (11) for σ_2 we find (the derivation is given in the appendix)

$$T = g\pi^3 \frac{\beta_2(s_{b\bar{e}})}{\sin \pi\alpha_{b\bar{e}}} \int_0^\infty ds_1\, \sigma_1(s_1, s_{a\bar{c}}) \int_0^1 \prod_{i=1}^4 d\alpha_i \frac{\delta\left(\sum_{i=1}^4 \alpha_i - 1\right)}{(d' + i\epsilon)^2}$$
$$\times (-\alpha_2 s_{de} - \alpha_4 s_{ab})^{\alpha_{b\bar{e}}} \,, \tag{15}$$

where d' is obtained from d by putting $\alpha_5 = 0$:

$$d' = \alpha_1\alpha_2 s_{a\bar{c}} + \alpha_1\alpha_3 s_{b\bar{e}} + \alpha_3\alpha_4 s_{cd} + \alpha_1\alpha_4 m_a^2 + \alpha_2\alpha_4 m_c^2$$
$$+ \alpha_2\alpha_3 m_d^2 - (\alpha_1 + \alpha_2 + \alpha_3)\mu^2 - \alpha_4 s_1 \,. \tag{16}$$

The structure of the amplitude T in eq. (15) is essentially that of the box diagram in fig. 6, the reggeon being treated as a scalar particle. The only difference is that the integrand is multiplied by a factor $(-\alpha_2 s_{de} - \alpha_4 s_{ab})^{\alpha_{b\bar{e}}}$, which describes the correlations due to the Reggeon spin. It is interesting to note that the amplitude (15) looks very similar to the DRM in this respect [17].

Consider now the limit $\kappa \to 0$. This implies $s_{de} \to 0$ in eq. (15), so that the extra factor in the integrand reduces to $(-\alpha_4 s_{ab})^{\alpha_{b\bar{e}}}$. The singularity structure of T is then determined by the zeroes of the denominator function d' in eq. (15). Hence the reggeon-particle amplitude in fig. 6 has the singularity structure of a normal four-point function.

We must still verify that the integral over s_1 in eq. (15) converges. The behaviour of $\sigma_1(s_1, s_{a\bar{c}})$ at large s_1 is given by eq. (11) and ensures convergence of the representation (15) when $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} > 0$. If $\alpha_{b\bar{e}} - \alpha_{a\bar{c}} < 0$ there is a singular term in T when $\kappa \to 0$, proportional to $(-\kappa)^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}}$. The singularity does not depend on s_{cd} and is therefore the same as the singularity in the double-Regge limit, eq. (3). As in the case of the DRM above we can subtract this singularity from the amplitude. The amplitude has then, for all values of $\alpha_{b\bar{e}} - \alpha_{a\bar{c}}$, only the singularities which come from the vanishing of the denominator function d' in eq. (15).

If $\kappa \neq 0$ it can readily be seen that the amplitude $\tilde{T}$, defined by eqs. (4) and (15), has singularities in s_{cd} which are not associated with zeroes of the denominator function d'. However, as in the case of the DRM these new singularities are not present in the leading term when $s_{cd} \to \infty$. In this limit the structure of T is given by eq. (3). The only singularities of $\tilde{T}$ (eq. (5)) are those of the first term in eq. (3), and correspond to normal singularities in s_{cd} (i.e. to zeroes of the denominator function).

If follows that also when $\kappa \neq 0$ the properties of the perturbation theory model (12) are similar to those of the DRM. An FESR can be derived, to which the new singularities contribute a term which does not depend on the cut-off. This term goes to zero in the limit $\kappa \to 0$.

4. Signature

Before we can write down the FESR we should construct amplitudes with definite signature in the $b\bar{e}$ and $a\bar{c}$ channels. Such amplitudes are most conveniently described using variables which are symmetric (or antisymmetric) under crossing. We shall begin be defining as set of such variables. We then discuss the effects of signature in the double Regge limit. Finally we use the DRM to study the properties of the signatured amplitude in the single-Regge limit.

4.1. Crossing-symmetric variables

In the double-Regge limit (fig. 3) all the large variables have simple properties under crossing. For example under line reversal in the $a\bar{c}$ channel (i.e. $a \leftrightarrow \bar{c}$) $s_{ab} = -s_{\bar{c}b}$. This is no longer true in the single Regge limit (fig. 1). The three large variables s_{ab}, $s_{\bar{c}b}$ and s_{de} are related through

$$s_{de} = s_{ab} + s_{\bar{c}b} \tag{17}$$

(we ignore terms like s_{cd}/s_{ab} which vanish in the single-Regge limit). From eq. (17) we can see that s_{de} is symmetric when $a \leftrightarrow \bar{c}$. Instead of s_{ab} we shall choose as our independent variable the combination σ, which is antisymmetric when $a \leftrightarrow \bar{c}$:

$$\sigma = \tfrac{1}{2}(s_{ab} - s_{\bar{c}b}) \,. \tag{18}$$

Both s_{dc} and σ are antisymmetric under $b \leftrightarrow \bar{e}$.

In analogy with four-point amplitudes we shall use the crossingodd variable ν,

$$\nu = (p_a + p_c) \cdot p_d = s_{cd} + \tfrac{1}{2}(s_{a\bar{c}} - s_{b\bar{e}} \cdot 2m_c^2 - m_d^2)\,, \tag{19}$$

to describe the reggeon-particle amplitude (instead of s_{cd}). Finally we define the variable

$$\kappa_s = -\frac{\upsilon s_{de}}{\sigma} \tag{20}$$

which is symmetric both under $a \leftrightarrow \bar{c}$ and $b \leftrightarrow \bar{e}$, and replaces $\kappa = s_{cd}s_{de}/s_{ab}$ used above.

4.2. The double-Regge limit

Let $T_{\tau_1\tau_2}$ be an amplitude with signature τ_1 in the $b\bar{e}$ channel and τ_2 in the $a\bar{c}$ channel. This amplitude can be constructed by adding four terms as in fig. 7, where a cross indicates that the reggeon line is to be twisted. The double-Regge limit of the amplitude in fig. 7a is given in eq. (1). The other amplitudes in fig. 7 are similar, except that they have left-hand cuts in some of the large variables. The full amplitude is then [13, 18, 19]

$$\begin{aligned} T_{\tau_1\tau_2} = {} & [(-\sigma)^{\alpha_{b\bar{e}}} + \tau_1(\sigma)^{\alpha_{b\bar{e}}}]\,[(-\upsilon)^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}} + \tau_1\tau_2(\upsilon)^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}}]\; V_1(s_{b\bar{e}}, s_{a\bar{c}}\,; \kappa) \\ & + [(-\sigma)^{\alpha_{a\bar{c}}} + \tau_2(\sigma)^{\alpha_{a\bar{c}}}]\,[(-s_{de})^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}} \\ & + \tau_1\tau_2(s_{de})^{\alpha_{b\bar{e}} - \alpha_{a\bar{c}}}]\; V_2(s_{a\bar{c}}, s_{b\bar{e}}\,; \kappa)\,. \end{aligned} \tag{21}$$

The structure of $T_{\tau_1\tau_2}$ in the double-Regge limit is similar to that of the amplitude T (eq. (1)). There are two terms in eq. (21), the first of which has cuts in σ and υ, and the second in σ and s_{de}. As before, only the first term in eq. (21) can be dual to resonances in υ. The second term may be eliminated either by taking the limit $\kappa_s \to 0$ (which implies $s_{de} \to 0$ in eq. (17)) or, if $\kappa_s \neq 0$, by defining a new amplitude $\tilde{T}_{\tau_1\tau_2}$.

Analogously to what was done in sect. 2, we define $\tilde{T}_{\tau_1\tau_2}$ by an analytic continuation in s_{de}. In the physical limit of the amplitude $T_{\tau_1\tau_2}$ all variables approach their cuts from above $(+i\epsilon)$. We denote this limit of $T_{\tau_1\tau_2}$ by $T_{\tau_1\tau_2}(s_{de} + i\epsilon)$. We now define the limit $T_{\tau_1\tau_2}(s_{de} - i\epsilon)$, where all variables approach their cuts from above except s_{de}, which approaches from below $(-i\epsilon)$. $T_{\tau_1\tau_2}(s_{de} - i\epsilon)$ can be obtained from the physical limit by continuig s_{de} along a circle, keeping σ and υ fixed. This is illustrated in fig. 8 for the case of a term with a righthand cut in s_{de}.

When continuing s_{de} we have to take care not to encircle the branch points at $s_{ab} = 0$ and $s_{\bar{c}b} = 0$, as these variables would then be evaluated in an unphysical

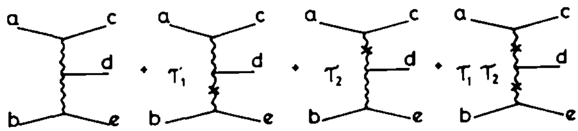

Fig. 7. The four terms which have to be added to obtain a definite signature τ_1 in the $b\bar{e}$ channel and τ_2 in the $a\bar{c}$ channel. A cross indicates that the reggeon line is to be twisted.

limit. It follows from eqs. (17) and (18) that this is ensured if $|s_{de}/\sigma| < 2$ during the continuation. This restriction is of course, only relevant in the single-Regge limit.

The definition of $\tilde{T}_{\tau_1\tau_2}$ is the same as that of $\tilde{T}$, given by eq. (4):

$$\tilde{T}_{\tau_1\tau_2} = \frac{1}{2i\sin\pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} [e^{i\pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} T_{\tau_1\tau_2}(s_{de} + i\epsilon)$$

$$- e^{-i\pi(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})} T_{\tau_1\tau_2}(s_{de} - i\epsilon)] \; . \tag{22}$$

In the double-Regge limit only the first term of $T_{\tau_1\tau_2}$ (eq. (21)) contributes to $\tilde{T}_{\tau_1\tau_2}$:

$$\tilde{T}_{\tau_1\tau_2} = [(-\sigma)^{\alpha_{b\bar{e}}} + \tau_1(\sigma)^{\alpha_{b\bar{e}}}]\,[(-\upsilon)^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}}$$

$$+ \tau_1\tau_2(\upsilon)^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}}]\; V_1(s_{b\bar{e}}, s_{a\bar{c}}; \kappa) \; . \tag{23}$$

We therefore expect that the υ-discontinuity of $\tilde{T}_{\tau_1\tau_2}$ in eq. (23) is dual to normal singularities (resonances) in υ.

4.3. The single-Regge limit

We shall now investigate the structure of $\tilde{T}_{\tau_1\tau_2}$ in the single-Regge limit (fig. 1) using the DRM. In this model $T_{\tau_1\tau_2}$ is a sum of four B_5 functions as in fig. 7. A twist on a reggeon line indicates that the ordering of two particles is to be reversed.

The amplitude for the diagram in fig. 7a is given in eq. (6), and the other three amplitudes can be obtained by replacing $a \leftrightarrow \bar{c}$, $b \leftrightarrow \bar{e}$. All amplitudes consist of two terms, of which only the first contributes to $\tilde{T}_{\tau_1\tau_2}$. The expression for $\tilde{T}_{\tau_1\tau_2}$ in the single-Regge limit is thus

$$\tilde{T}_{\tau_1\tau_1} = \Gamma(-\alpha_{b\bar{e}})[(-\sigma)^{\alpha_{b\bar{e}}} + \tau_1(\sigma)^{\alpha_{b\bar{e}}}]$$

$$\times \left[\left(1 + \frac{1}{2}\frac{\kappa_s}{\upsilon}\right)^{\alpha_{b\bar{e}}} B_4(-\alpha_{cd}, -\alpha_{a\bar{c}} + \alpha_{b\bar{e}}) F\left(-\alpha_{b\bar{e}}, -\alpha_{cd}; 1 - \alpha_{b\bar{e}} + \alpha_{a\bar{c}}; \frac{\kappa_s/\upsilon}{1 + \frac{1}{2}\kappa/\upsilon}\right)\right.$$

$$+ \tau_1\tau_2 \left(1 - \frac{1}{2}\frac{\kappa_s}{\upsilon}\right)^{\alpha_{b\bar{e}}} B_4(-\alpha_{a\bar{d}}, -\alpha_{a\bar{c}} + \alpha_{b\bar{e}}) F\left(-\alpha_{b\bar{e}}, -\alpha_{a\bar{d}}; 1 - \alpha_{b\bar{e}} + \alpha_{a\bar{c}}; -\frac{\kappa_s/\upsilon}{1 - \frac{1}{2}\kappa_s/\upsilon}\right),$$

$$\tag{24}$$

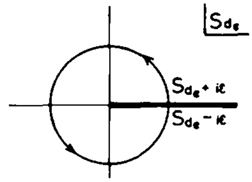

Fig. 8. The path of continuation in s_{de} used in the definition of $T_{\tau_1\tau_2}(s_{\mathrm{de}}-i\epsilon)$. The singularity structure is that of an amplitude with a right-hand cut in s_{de}. Note that the branch point of an amplitude with a left-hand cut in s_{de} is similarly encircled.

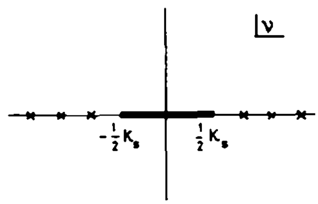

Fig. 9. The singularity structure in s_{cd} of the signatured amplitude $\tilde{T}_{\tau_1\tau_2}$ in the dual resonance model. Crosses correspond to poles and the thick line to a cut.

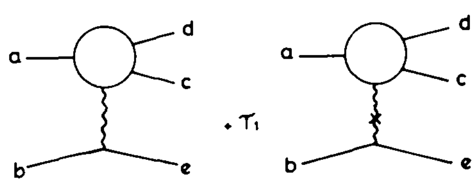

Fig. 10. The two B_5 functions given by eq. (25) in the text.

If in eq. (24) we let $\kappa_s \to 0$ we find that $\tilde{T}_{\tau_1\tau_2}$ reduces to a sum of B_4 functions. The only singularities of $\tilde{T}_{\tau_1\tau_2}$ in υ are then the resonance poles. When $\kappa_s \neq 0$ $\tilde{T}_{\tau_1\tau_2}$ has, in addition, a cut in υ for $-\frac{1}{2}\kappa_s \leqslant \upsilon \leqslant \frac{1}{2}\kappa_s$ (see fig. 9). Hence the properties of the signatured amplitude are very similar to those of the non-signatured amplitude that we discussed in the previous section.

There are two more B_5 functions which contribute to the single-Regge limit, in addition to the four shown in fig. 7. They are shown in fig. 10. Denoting their combined contribution by $B_5(s,u)$ we have

$$B_5(s,u) = \Gamma(-\alpha_{\mathrm{b}\bar{\mathrm{e}}})[(-\sigma)^{\alpha_{\mathrm{b}\bar{\mathrm{e}}}} + \tau_1 (\sigma)^{\alpha_{\mathrm{b}\bar{\mathrm{e}}}}] \left(1 + \frac{1}{2}\frac{\kappa_s}{\upsilon}\right)^{\alpha_{\mathrm{b}\bar{\mathrm{e}}}}$$

$$\times B_4(-\alpha_{\mathrm{cd}}, -\alpha_{\mathrm{a}\bar{\mathrm{d}}}) F\left(-\alpha_{\mathrm{b}\bar{\mathrm{e}}}, -\alpha_{\mathrm{cd}}; -\alpha_{\mathrm{cd}} - \alpha_{\mathrm{a}\bar{\mathrm{d}}}; \frac{\kappa_s/\upsilon}{1 + \frac{1}{2}\kappa_s/\upsilon}\right). \tag{25}$$

The singularity structure of $B_5(s,u)$ is the same as that of $\tilde{T}_{\tau_1\tau_2}$ in eq. (24) and shown in fig. 9. $B_5(s,u)$ is symmetric under a $\leftrightarrow$ $\bar{\mathrm{c}}$ and vanishes exponentially [19] in the double-Regge limit. It must therefore be superconvergent.

We have now investigated all amplitudes that contribute to the single-Regge limit in the DRM. The properties of the amplitudes with definite signature in the b$\bar{\mathrm{e}}$ and a$\bar{\mathrm{c}}$ channels are completely analogous to the properties of the amplitude with only right-hand singularities, discussed in sect. 3. The conclusions about the singularity structure which we reached in that section are therefore valid for the full amplitude with right- and left-hand singularities.

A similar analysis can be done using the perturbation theory model described in sect. 3. The conclusions reached are the same as for the DRM.

5. The finite-energy sum rules

We define the reggeon-particle amplitude $f_i(\upsilon, s_{\mathrm{a}\bar{\mathrm{c}}}, s_{\mathrm{b}\bar{\mathrm{e}}}, \kappa_s)$ for the process a + i → c + d (fig. 1) by the relation

$$\tilde{T}_i = \beta^i_{\mathrm{b}\bar{\mathrm{e}}}(s_{\mathrm{b}\bar{\mathrm{e}}}) \frac{\tau_i + \mathrm{e}^{-i\pi\alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}})}}{\sin \pi\alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}})} \sigma^{\alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}})} f_i\,. \tag{26}$$

In eq. (26), $\tilde{T}_i$ is the amplitude related as in eq. (22) to the amplitude T_i for the process a + b → c + d + e, the reggeon i being exchanged in the b$\bar{\mathrm{e}}$ channel (fig. 1). $\beta^i_{\mathrm{b}\bar{\mathrm{e}}}$ is the reggeon-particle-particle vertex function, and the definition of the variables σ, υ and κ_s is given in eqs. (18), (19) and (20).

From our discussion above we expect that f_i, as a function of υ, has the normal singularities of a four-point function when $\kappa_s = 0$. If $\kappa_s \neq 0$ there are additional singularities which do not, however, contribute to the limit $|\upsilon| \to \infty$ of f_i. In the DRM these additional singularities take the form of a cut $-\frac{1}{2}\kappa_s \leqslant \upsilon \leqslant \frac{1}{2}\kappa_s$ (see fig. 9).

According to (23) and 26) the behaviour of f_i as $\upsilon \to \infty$ is

$$f_i = \sum_j \beta^j_{\mathrm{a}\bar{\mathrm{c}}}(s_{\mathrm{a}\bar{\mathrm{c}}}) \frac{\tau_i\tau_j + \exp\left[-i\pi(\alpha_j(s_{\mathrm{a}\bar{\mathrm{c}}}) - \alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}}))\right]}{\sin \pi\left[\alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}}) - \alpha_j(s_{\mathrm{a}\bar{\mathrm{c}}})\right]}$$

$$\times\, \upsilon^{\alpha_j(s_{\mathrm{a}\bar{\mathrm{c}}}) - \alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}})} V^{\mathrm{d}}_{ij}(s_{\mathrm{b}\bar{\mathrm{e}}}, s_{\mathrm{a}\bar{\mathrm{c}}}; \kappa)\,, \tag{27}$$

where V_{ij}^{d} is the part of the reggeon (i) – reggeon (j) – particle (d) vertex which is multiplied by v (i.e. V_1 in eq. (21)).

The FESR for the amplitude f_i can now be derived in the standard way [3, 4]. They are

$$c_i^{(n)}(s_{\mathrm{b}\bar{\mathrm{e}}}, s_{\mathrm{a}\bar{\mathrm{c}}}; \kappa_s) + \int_{v_0}^{N} \mathrm{d}\nu\, \nu^n\, \mathrm{Im}\,[f_i(\nu + i\epsilon) + (-1)^{n+1} f_i(-\nu - i\epsilon)]$$

$$= \sum_j [1 + (-1)^{n+1}\tau_i\tau_j]\, \beta^j_{\mathrm{a}\bar{\mathrm{c}}}(s_{\mathrm{a}\bar{\mathrm{c}}})\, V_{ij}^{\mathrm{d}}(s_{\mathrm{b}\bar{\mathrm{e}}}, s_{\mathrm{a}\bar{\mathrm{c}}}; \kappa_s)$$

$$\times \frac{N^{\alpha_j(s_{\mathrm{a}\bar{\mathrm{c}}}) - \alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}}) + n + 1}}{\alpha_j(s_{\mathrm{ac}}) \quad \alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}}) + n + 1}, \tag{28}$$

where

$$\mathrm{Im}\, f_i(\nu \pm i\epsilon) = \frac{1}{2i}[f_i(\nu \pm i\epsilon, s_{\mathrm{a}\bar{\mathrm{c}}}, s_{\mathrm{be}}, \kappa_s) - f_i(\nu \mp i\epsilon, s_{\mathrm{a}\bar{\mathrm{c}}}, s_{\mathrm{b}\bar{\mathrm{e}}}, \kappa_s)] \,.$$

The integral in eq. (28) is over the normal four-point singularities of f_i (i.e. pole terms, resonances, etc.). The additional singularities of f_i contribute the term $c_i^{(n)}$.

As discussed in the previous sections, $c_i^{(n)}(s_{\mathrm{b}\bar{\mathrm{e}}}, s_{\mathrm{a}\bar{\mathrm{c}}}; \kappa_s)$ vanishes as $\kappa_s \to 0$. In the DRM it is readily seen that

$$c_i^{(n)}(s_{\mathrm{b}\bar{\mathrm{e}}}, s_{\mathrm{a}\bar{\mathrm{c}}}; \kappa_s) \propto (\kappa_s)^{n+1} \qquad \text{as} \qquad \kappa_s \to 0 \,. \tag{29}$$

For small κ_s the higher moment sum rules are thus less sensitive to the unknown term $c_i^{(n)}$.

6. Applications

At present the only way of obtaining $\mathrm{Im}\, f_i(\nu)$ in eq. (28) from experimental data is to assume that the absorptive part is dominated by the resonance contributions. The consistency of eq. (28) with the data can then be tested by varying the cut-off N. This should be done at several values of κ_s and n, so that the restriction (29) can be applied.

Such an application of the FESR is analogous to a recent analysis [8] of quasi-two-body reactions using the inclusive FMSR. However, it should give considerably more information, as not only the total production cross section of the resonances but also their decay distributions can be correlated. In addition, one can avoid certain resonances whose production mechanisms are not clearly understood (e.g., Q may contribute to K + f → anything but not to K + f → K + π).

In general the two parts of the double-Regge vertex have to be generated separately by summing over resonances in s_{cd} and in s_{de} (and their crossed channels). In some applications, however, the two parts can be related to each other by exchange-degeneracy arguments. This happens in particular when particle d is a π-meson and the reggeons exchanged are any of the four meson trajectors f-ρ-ω-A_2. As we think this case may be of practical interest we shall discuss it in some detail here.

Consider the system $\mathrm{abc}\overline{\mathrm{de}} = K^-K^+K^+\pi^-\overline{K}^0$ in the double-Regge limit as in fig. 11a (all particles are treated as incoming). As in sect. 4 we shall assume that the contribution of a given exchange $(\alpha_i(s_{\mathrm{b}\bar{\mathrm{e}}}), \alpha_j(s_{\mathrm{a}\bar{\mathrm{c}}}))$ to the amplitude T is of the form

$$T(i,j) = [(-\sigma)^{\alpha_{\mathrm{b}\bar{\mathrm{e}}}} + \tau_i \sigma^{\alpha_{\mathrm{b}\bar{\mathrm{c}}}}]\,[(-\nu)^{\alpha_{\mathrm{a}\bar{\mathrm{c}}} - \alpha_{\mathrm{b}\bar{\mathrm{c}}}} + \tau_i\tau_j{}^{\alpha_{\mathrm{a}\bar{\mathrm{c}}} - \alpha_{\mathrm{b}\bar{\mathrm{e}}}}]\, V_1(i,j)$$

$$+ [(-\sigma)^{\alpha_{\mathrm{a}\bar{\mathrm{c}}}} + \tau_j \sigma^{\alpha_{\mathrm{a}\bar{\mathrm{c}}}}]\,[(-s_{\mathrm{de}})^{\alpha_{\mathrm{b}\bar{\mathrm{c}}} - \alpha_{\mathrm{a}\bar{\mathrm{c}}}} + \tau_i\tau_j s_{\mathrm{de}}^{\alpha_{\mathrm{b}\bar{\mathrm{c}}} - \alpha_{\mathrm{a}\bar{\mathrm{c}}}}]\, V_2(j,i)\,. \tag{30}$$

The full amplitude is

$$T = T(A_2^+, \mathrm{f}) + T(A_2^+, \rho^0) + T(\rho^+, \omega) + T(\rho^+, A_2^0)\,. \tag{31}$$

By drawing the duality diagrams it is easy to see that the only planar diagram is the one where the particles a, b, c, d, e are ordered as in fig. 11a. This means that the full amplitude T should have only a right-hand cut in each of the variables σ, ν and s_{de}. Hence six of the eight terms in eq. (30) have to cancel in the sum (31). This gives six relations between the vertex functions $V_k(i,j)$, $k = 1,2$.

Six further relations can be obtained by considering the system $\mathrm{abc}\overline{\mathrm{de}} = K^+K^-\overline{K}^0\pi^-K^+$ (fig. 11b). Combined with isospin invariance these relations imply that all non-zero vertex functions $V_k(i,j)$ are degenerate ($k = 1, 2$):

$$V_k(A_2^+, \mathrm{f}) = V_k(\rho^+, \omega) = V_k(A_2^+, \rho^0)$$

$$= -V_k(\mathrm{f}, A_2^+) = -V_k(\omega, \rho^+) = -V_k(\rho^0, A_2^+)\,, \qquad k = 1, 2$$

Finally, observing that the process in fig. 11a is identical to the one in fig. 11b,

$$V_1(A_2^+, \mathrm{f}) = V_2(A_2^+, \mathrm{f})\,.$$

It follows that all the vertex functions are related.

The degeneracy of the vertex functions means that the full double-Regge vertex can be obtained by summing the resonances in only one system, e.g. in s_{cd}. Consistency with the sum of resonance in the other system (s_{de}) then requires that the two sets of resonances must be related. These predictions make the application of the FESR particularly interesting to reactions where f, ρ, ω or A_2 are the dominating exchanges.

We are grateful to all our colleagues for helpful discussions. One of us (J.K.) is indebted to Dr. R.J.N. Phillips for his kind hospitality at the Theory Division of the Rutherford Laboratory, where this work has been done.

Appendix

In this appendix we shall derive the explicit expressions for the perturbation theory amplitude T in the single- and double-Regge limits. The definition of T in the non-asymptotic region is given by eqs. (13) and (14). We shall assume that $-1 < \alpha_{b\bar{e}}, \alpha_{a\bar{c}} < 0$. It is straightforward to continue the expressions to arbitrary values of the momentum transfers.

In the single Regge limit $s_{ab} \to -\infty$, $s_{de} \to -\infty$ while s_{de}/s_{ab}, s_{cd}, $s_{b\bar{e}}$ and $s_{a\bar{c}}$ remain fixed. The leading contribution to the integral in eq. (13) comes from large s_2. Substituting the leading behaviour of σ_2,

$$\sigma_2(s_2, s_{b\bar{e}}) \simeq \beta_2(s_{b\bar{e}}) s_2^{\alpha_{b\bar{e}}}, \qquad s_2 \to \infty, \tag{A.1}$$

in eq. (13), the integral over s_2 can be explicitly done. We get

$$T = g\pi^3 \beta_2(s_{b\bar{e}})\alpha_{b\bar{e}}(\alpha_{b\bar{e}} - 1)\frac{e^{-i\pi\alpha_{b\bar{e}}}}{\sin \pi\alpha_{b\bar{e}}}$$

$$\times \int_0^\infty \sigma_1(s_1, s_{a\bar{c}})\, ds_1 \int_0^1 \prod_{i=1}^5 d\alpha_i \frac{\delta\left(\sum_{i=1}^5 \alpha_i - 1\right) \alpha_5^{-\alpha_{b\bar{e}}-1}}{[d'' + i\epsilon]^{2-\alpha_{b\bar{e}}}}, \tag{A.2}$$

where

$$d'' = \alpha_5(\alpha_2 s_{de} + \alpha_4 s_{ab}) + \alpha_1\alpha_2 s_{a\bar{c}} + \alpha_1\alpha_2 s_{b\bar{e}}$$
$$+ \alpha_3\alpha_4 s_{cd} + \alpha_1\alpha_4 m_a^2 + \alpha_2\alpha_4 m_c^2 + \alpha_2\alpha_3 m_d^2 + \alpha_3\alpha_5 m_e^2$$
$$+ \alpha_1\alpha_5 m_b^2 - (\alpha_1 + \alpha_2 + \alpha_3)\mu^2 - \alpha_4 s_1\,. \tag{A.3}$$

Because the large variables s_{de} and s_{ab} both are multiplied by α_5 in eq. (A.3), the leading contribution to T comes from small α_5. If we scale α_5,

$$\alpha_5 = -\frac{x}{\alpha_2 s_{de} + \alpha_4 s_{ab}}, \tag{A.4}$$

the integral over x can be extended from zero to infinity. The expression for T is then

$$T = g\pi^3\beta_2(s_{b\bar{e}})\alpha_{b\bar{e}}(\alpha_{b\bar{e}} - 1)\frac{e^{-i\pi\alpha_{b\bar{e}}}}{\sin \pi\alpha_{b\bar{e}}}\int_0^\infty ds_1\, \sigma_1(s_1, s_{a\bar{c}})$$

$$\times \int_0^1 \prod_{i=1}^4 d\alpha_i \delta\left(\sum_{i=1}^4 \alpha_i - 1\right)(-\alpha_2 s_{de} - \alpha_4 s_{ab})^{\alpha_{b\bar{e}}}\int_0^\infty dx \frac{x^{-\alpha_{b\bar{e}}-1}}{[d' - x + i\epsilon]^{2-\alpha_{b\bar{e}}}}\,, \tag{A.5}$$

where the expression for d' is given in eq. (16). The integral over x in (A.5) can be done explicitly and we then obtain the expression (15) for T in the single-Regge limit.

Next consider the double-Regge limit. We have to let $s_{cd} \to \infty$ keeping $s_{a\bar{c}}$, $s_{b\bar{e}}$ and $\kappa = s_{cd}s_{de}/s_{ab}$ fixed in the expression (15) for T. Again, the dominant contribution comes from large s_1. Substituting the Regge behaviour (11) of σ_1 we get

$$T = -g\pi^4 \frac{\beta_1(s_{a\bar{c}})\beta_2(s_{b\bar{e}})}{\sin \pi\alpha_{a\bar{c}} \sin \pi\alpha_{b\bar{e}}} e^{-i\pi\alpha_{a\bar{c}}}\alpha_{a\bar{c}}\int_0^1 \prod_{i=1}^4 d\alpha_i \delta\left(\sum_1^4 \alpha_i - 1\right)$$

$$\times \frac{\alpha_4^{-\alpha_{a\bar{c}}-1}}{(d''' + i\epsilon)^{1-\alpha_{a\bar{c}}}}(-\alpha_2 s_{de} - \alpha_4 s_{ab})^{\alpha_{b\bar{e}}}\,, \tag{A.6}$$

where

$$d''' = \alpha_3\alpha_4 s_{cd} + \alpha_1\alpha_2 s_{a\bar{c}} + \alpha_1\alpha_3 s_{b\bar{e}} + \alpha_1\alpha_4 m_a^2$$
$$+ \alpha_2\alpha_4 m_c^2 + \alpha_2\alpha_3 m_d^2 - (\alpha_1 + \alpha_2 + \alpha_3)\mu^2\,. \tag{A.7}$$

If we define the new integration variable z by

$$\alpha_4 = \frac{\alpha_2\kappa}{s_{cd}} z\,, \tag{A.8}$$

we get for the leading term in T,

$$T = -g\pi^4 \frac{\beta_1(s_{a\bar{c}})\beta_2(s_{b\bar{e}})}{\sin \pi\alpha_{a\bar{c}} \sin \pi\alpha_{b\bar{e}}}(-s_{ab})^{\alpha_{a\bar{c}}}(-s_{de})^{\alpha_{b\bar{e}}-\alpha_{a\bar{c}}} e^{-i\pi\alpha_{a\bar{c}}}\alpha_{a\bar{c}}$$

$$\times \int_0^1 \prod_{i=1}^3 d\alpha_i \delta\left(\sum_{i=1}^3 \alpha_i - 1\right)\alpha_2^{\alpha_{b\bar{e}}-\alpha_{a\bar{c}}}\int_0^\infty dz\, z^{-\alpha_{a\bar{c}}-1}\frac{(1+z)^{\alpha_{b\bar{e}}}}{(\bar{d} + \alpha_2\alpha_3\kappa z + i\epsilon)^{1-\alpha_{a\bar{c}}}}\,, \tag{A.9}$$

where

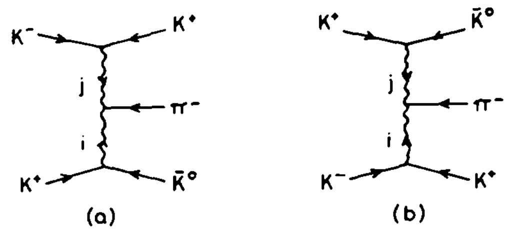

Fig. 11. (a) The double-Regge limit of the process $K^- + K^+ \to K^- + \pi^+ + K^0$. All particles are treated as incoming. (b) The same reaction as in fig. 11a, but with a change in the particle ordering: $a \longleftrightarrow b$, $c \longleftrightarrow e$.

$$\bar{d} = \alpha_1\alpha_2 s_{a\bar{c}} + \alpha_1\alpha_3 s_{be} + \alpha_2\alpha_3 m_d^2 - (\alpha_1 + \alpha_2 + \alpha_3)\mu^2 \tag{A.10}$$

The denominator $(\bar{d} + \alpha_2\alpha_3\kappa z + i\epsilon)^{1-\alpha_{a\bar{c}}}$ in (A9) can be changed into an exponential using the formula

$$\frac{1}{(z+i\epsilon)^\mu} = \frac{e^{-i\pi\mu}}{\Gamma(\mu)} \int_0^\infty e^{\lambda(z+i\epsilon)}\lambda^{\mu-1}\,d\lambda\,. \tag{A.11}$$

The resulting expression for T can then be expressed in terms of the confluent hypergeometric function* $\psi(a, b; x)$. This function can be written [20] as a sum of two entire functions $\phi(a, b; x)$, which establishes the structure (1) of T in the double-Regge limit. The explicit expression for the vertex function V_1 in eq. (1) is

$$V_1(s_{b\bar{e}}, s_{a\bar{c}}; \kappa) = -g\pi^4 \frac{\beta_1(s_{a\bar{c}})\beta_2(s_{b\bar{e}})}{\sin\pi\alpha_{a\bar{c}}\sin\pi\alpha_{b\bar{e}}} \frac{\Gamma(\alpha_{b\bar{e}} - \alpha_{a\bar{c}})}{\Gamma(-\alpha_{a\bar{c}})}$$

$$\times \int_0^1 \prod_{i=1}^3 d\alpha_i\,\delta\left(\sum_{i=1}^3 \alpha_i - 1\right) \alpha_3^{\alpha_{a\bar{c}} - \alpha_{b\bar{e}}} \int_0^\infty d\lambda\,\lambda^{-\alpha_{b\bar{e}}}\,e^{\lambda(\bar{d}+i\epsilon)}$$

$$\times\,\phi(-\alpha_{b\bar{e}}, \alpha_{a\bar{c}} - \alpha_{b\bar{e}} + 1; -\alpha_2\alpha_3\lambda\kappa). \tag{A.12}$$

The amplitude T being symmetric, V_1 and V_2 are the same functions in this model.

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